



PAPER

The SUSY partners of the QES sextic potential revisited

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Abstract

In this paper, the SUSY partner Hamiltonians of the quasi-exactly solvable (QES) sextic potential $V^{\text{qes}}(x) = \nu x^6 + 2\nu\mu x^4 + [\mu^2 - (4N + 3)\nu]x^2$, $N \in \mathbb{Z}^+$, are revisited from a Lie algebraic perspective. It is demonstrated that, in the variable $z = x^2$, the underlying $\mathfrak{sl}_2(\mathbb{R})$ hidden algebra of $V^{\text{qes}}(x)$ is inherited by its SUSY partner potential $V_1(x)$ only for $N = 0$. At fixed $N > 0$, the algebraic polynomial operator $h(x, \partial_x; N)$ that governs the N exact eigenpolynomial solutions of V_1 is derived explicitly. These odd-parity solutions appear in the form of zero modes. The potential V_1 can be represented as the sum of a polynomial and rational parts. In particular, it is shown that the polynomial component is given by V^{qes} with a different non-integer (cohomology) parameter $N_1 = N - \frac{3}{2}$. A confluent second-order SUSY transformation is also implemented for a modified QES sextic potential possessing the energy reflection symmetry. By taking N as a continuous real constant and using the Lagrange-mesh method, highly accurate values (~ 20 s. d.) of the energy $E_n = E_n(N)$ in the interval $N \in [-1, 3]$ are calculated for the three lowest states $n = 0, 1, 2$ of the system. The critical value N_c above which tunneling effects (instanton-like terms) can occur is obtained as well. At $N = 0$, the non-algebraic sector of the spectrum of V^{qes} is described by means of compact physically relevant trial functions. These solutions allow us to determine the effects in accuracy when the first-order SUSY approach is applied on the level of approximate eigenfunctions.

1. Introduction

In quantum mechanics, quasi-exactly solvable (QES) systems are spectral problems, $\mathcal{H}\psi = E\psi$, for which it is possible to obtain in closed (analytical) form only a finite number of exact solutions while the remaining ones remain unknown. A systematic approach to quasi-exact solvability is purely algebraic [1–3]. In this case, the Hamiltonian can be rewritten as a constant coefficient quadratic combination in the generators of a Lie algebra. The underlying hidden algebraic structure then leads to non-trivial dynamical features of these systems. For instance, one can mention the appearance of the so-called energy-reflection (ER) symmetry [4] as well as the existence of a generating function for a set of orthogonal polynomials $P_n(E)$ in the energy variable E [5].

Interestingly, starting from a given QES model, the generation of new classes of QES problems was pointed out by means of supersymmetric quantum mechanics (SUSY) techniques [6–9]. A key element is the factorization method as described first in Dirac's book [10] and further developed in [11], whereas complete reviews on supersymmetric quantum mechanics can be found in [12–16]. In particular, in [6], the SUSY framework is one of the three methods employed to derive new QES potentials. Even more, in the last two decades, much attention has been paid to quasi-solvable models from the point of view of \mathcal{N} -fold supersymmetry [17, 18] (see also [19] and references therein).

In most cases, the key ideas about QES models are illustrated taking as a prototype the sextic potential $V^{\text{qes}}(x) = \nu^2 x^6 + 2\nu\mu x^4 + (\mu^2 - (4N + 3)\nu)x^2$. This one-dimensional problem possesses a hidden \mathfrak{sl}_2 Lie algebra. If the parameter N is a positive integer number, it is possible to find only $(N + 1)$ exact analytical solutions. The exact ground state $\psi_0^{\text{qes}}(x; N)$ can always be obtained explicitly. Therefore, the first-order SUSY

partner Hamiltonian \mathcal{H}_1 with potential $V_1(x)$ can also be constructed for any positive integer value of N . A natural question arises, namely, Does the SUSY partner potential of V^{qes} possess a hidden Lie algebra?. To the best of the authors' knowledge, such a relevant question has not been previously studied in the literature. In this respect, the situation is very different from previous works where the emphasis relies on finding methods for generating QES potentials. Here, we focus on the search for a hidden algebraic structure and special properties of the first-order partner Hamiltonian \mathcal{H}_1 and its solutions.

Along these lines, we also explore the sextic QES system combining the SUSY framework and the variational method to analyze the connection, in the non-algebraic sector of the spectrum, between the bosonic and fermionic QES Hamiltonians. Since highly accurate approximate solutions are constructed, the description of the first-order SUSY mechanism at the level of approximate solutions is investigated in detail.

The goal of the present study is threefold. Firstly, for the lowest values of the parameter $N = 0, 1, 2$, we will derive an algebraic polynomial operator $h(x, \partial_x; N)$. As a distinguished feature, the zero modes of $h(x, \partial_x; N)$ give the N exact polynomial eigenfunctions of the SUSY partner potential $V_1(x)$ which, in general, is not a polynomial but a rational function in the variable x^2 . Towards a Lie-algebraic characterization, the analytical properties of V_1 are clearly indicated. For $N > 2$, the generic expression of $h(x, \partial_x; N)$ is presented as well. To overcome the 1-SUSY requirement of the explicit knowledge of the exact (analytical) ground state function $\psi_0^{\text{qes}}(x; N)$ a confluent second-order SUSY transformation is applied on a QES sextic potential to generate isospectral QES models.

Secondly, for the three lowest states $n = 0, 1, 2$ of the QES potential $V^{\text{qes}}(x)$, highly accurate values of the energy $E_n = E_n(N)$ as a function of N are displayed within the interval $N \in [-1, 3]$. Thus, we investigate the system with N being considered as a continuous real parameter. The numerical results are presented with ~ 20 significant digits; the corresponding calculations are performed using the Lagrange-mesh method. In particular, the critical value N_c above which tunneling effects can occur is computed.

Finally, for the case $N = 0$ where solely the exact ground state of $V^{\text{qes}}(x)$ is known, we generate approximate solutions for $V_1(x)$. They are constructed by means of SUSY techniques allied with the variational method. As a first step, compact variational trial functions ψ_{trial} for the excited states of V^{qes} are designed. They encode relevant physical properties of the system. Afterward, a SUSY transformation acting on these ψ_{trial} will produce approximate eigenfunctions for V_1 . This simple idea allows us to determine the effects in accuracy when SUSY is implemented on the level of approximate eigenfunctions, another interesting practical aspect absent in the literature.

2. Generalities

We consider the following one-dimensional spectral problem in non-relativistic quantum mechanics:

$$\mathcal{H} \psi(x) = E \psi(x), \quad \psi(x) \in \mathcal{L}^2, \quad (1)$$

defined on the real line, $x \in (-\infty, \infty)$. The corresponding Hamiltonian operator for our specific problem is of the form:

$$\mathcal{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} V^{\text{qes}}(x), \quad (2)$$

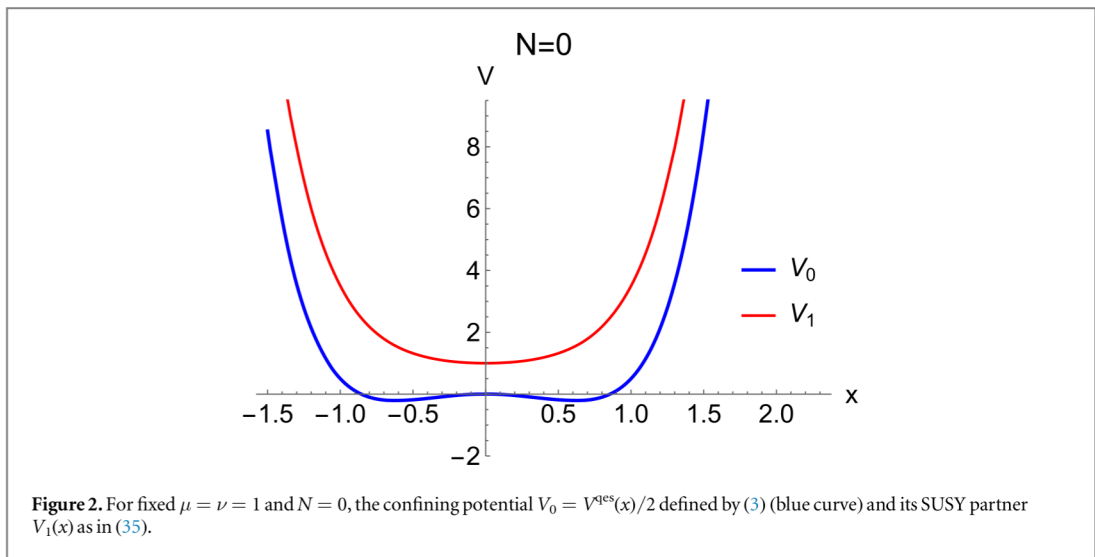
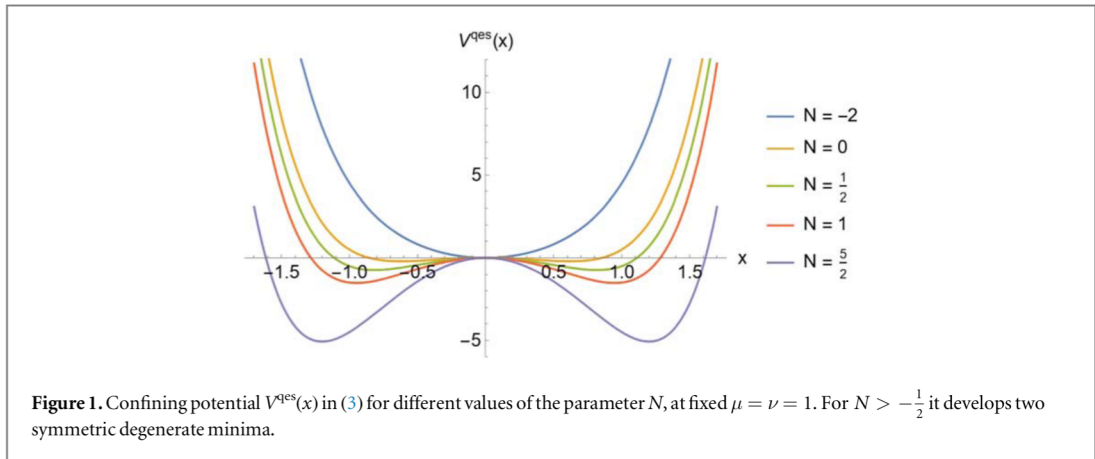
where m denotes the mass of the particle, and

$$V^{\text{qes}}(x) = \nu^2 x^6 + 2\nu\mu x^4 + [\mu^2 - (4N + 3)\nu] x^2, \quad (3)$$

is the quasi-exactly solvable sextic potential, here ν, μ, N are real parameters. At $\nu = 0$, the potential (3) reduces to the exactly-solvable simple harmonic oscillator $V^{\text{qes}}(x) = \mu^2 x^2$. Hereafter, we will adopt atomic units $\hbar = 1, m = 1$.

Some remarks are in order:

- For $\nu \neq 0$ and arbitrary N and μ, N not necessarily an integer number, the potential V^{qes} (3) admits infinitely many bound states, see figure 1. No scattering states occur.
- At fixed μ and $\nu > 0$, there exists a special value $N = \frac{\mu^2 - 3\nu}{4\nu}$ above which the potential (3) develops two symmetric degenerate minima. Thus, tunneling effects (instanton-like terms) can take place. For large $N \rightarrow \infty$, these two minima are located at $x_{\pm} \approx \pm \sqrt{2} \left(\frac{N}{3\nu}\right)^{1/4}$, respectively, and $V^{\text{qes}}(x_{\pm}) \approx \frac{4}{9}N(3\mu - 2\sqrt{3\nu N})$.
- The Hamiltonian (2) is invariant under the parity symmetry $x \rightarrow -x$, which gives rise to alternating symmetric and anti-symmetric bound states.



- At $\mu = 0$, N and $\nu \neq 0$ arbitrary, the energy reflection (ER) symmetry in (2) emerges [4]. For instance, the wave functions of the ER symmetric levels are connected to each other by analytic continuation $x \rightarrow ix$, $\psi_E \rightarrow \psi_{-E}$. In recent papers [20, 21] the energy reflection symmetry has been derived for $\mu \neq 0$.
- Formally, the most general eigenfunction of (2) reads [22, 23]

$$\begin{aligned} \psi(x) = e^{\frac{1}{4}(\nu x^4 + 2\mu x^2)} \times & \left(c_1 w \left[\frac{1}{4}(-2E - \mu), \nu \left(N + \frac{3}{2} \right), \frac{1}{2}, \mu, \nu, x^2 \right] \right. \\ & \left. + c_2 x w \left[\frac{1}{4}(-2E - 3\mu), \nu(N + 2), \frac{3}{2}, \mu, \nu, x^2 \right] \right), \end{aligned} \quad (4)$$

here c_1, c_2 are constants of integration, and $w(z) = w(q, \alpha, \gamma, \delta, \epsilon, z)$ satisfies the bi-confluent Heun differential equation $z w'' + w'(z^2 \epsilon + \delta z + \gamma) + w(\alpha z - q) = 0$. We will focus on the case where $\psi(x)$ is normalizable, and one of its factors is a polynomial function in the x – variable. Other solutions are also interesting; see for example [22] where a sextic potential with a centrifugal barrier is considered.

- The Hamiltonian (2) possesses a hidden $\mathfrak{sl}_2(\mathbb{R})$ Lie algebra [2]. In the particular case when N is a non-negative integer number, $N \in \mathbb{Z}^+$, this algebra admits a finite-dimensional irreducible representation and, thus, one can find algebraically only $(N + 1)$ exact even-parity eigenfunctions and their corresponding energies explicitly.
- As stated in [24], for a positive integer number N , the system (3) is intimately related to the exact cancellation of real and complex nonperturbative saddles to all orders in the semiclassical expansion.
- Finally, at $N = -j - \frac{1}{2}$, j a positive integer, we arrive to the sextic \mathcal{PT} - symmetric QES potential [25].

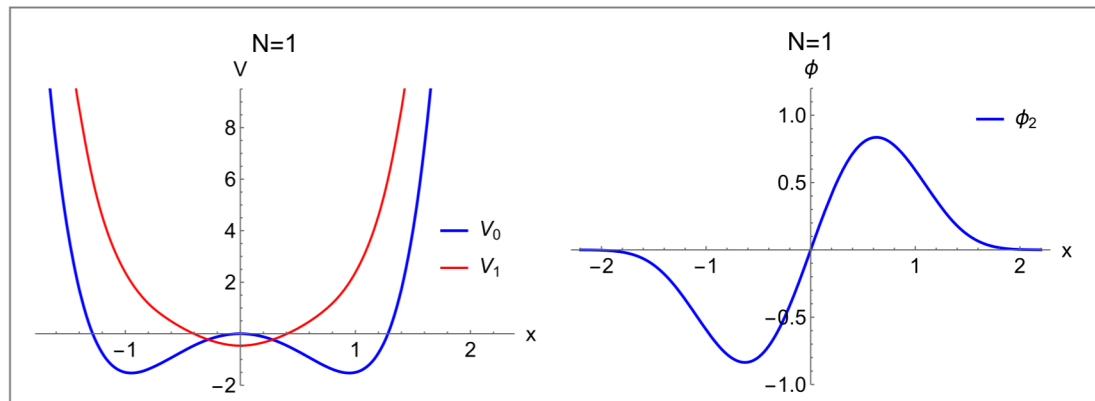


Figure 3. Left: For fixed $\mu = \nu = 1$, the confining potential $V_0 = V^{\text{qes}}(x)/2$ in (3) at $N = 1$ (blue curve) and its SUSY partner V_1 (red line) as in (36). Right: The (exact) first excited eigenfunction of \mathcal{H}_1 .

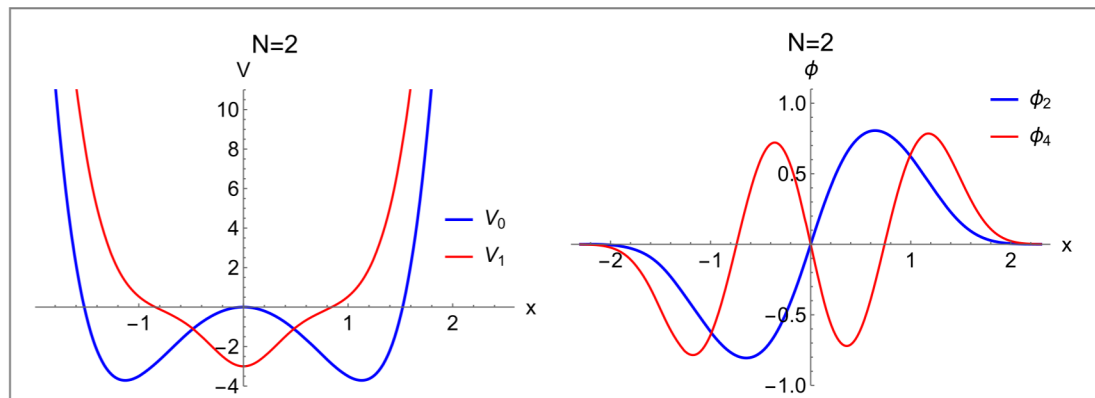


Figure 4. Left: For fixed $\mu = \nu = 1$, the confining potential $V_0 = V^{\text{qes}}(x)/2$ in (3) at $N = 2$ (blue curve) and its SUSY partner V_1 (red line) as in (40). Right: The (exact) first and third excited eigenfunctions of \mathcal{H}_1 .

3. The hidden \mathfrak{sl}_2 Lie algebra of $V^{\text{qes}}(x)$

We begin with the well-established result that the Hamiltonian \mathcal{H} in (2) can be transformed into a suitable \mathfrak{sl}_2 Lie algebraic operator [2, 26]. To this end, the gauge factor:

$$\Gamma(x) = \exp \left[-\frac{\nu}{4} x^4 - \frac{\mu}{2} x^2 \right], \tag{5}$$

is introduced. In the \mathbb{Z}_2 -invariant variable $z = x^2$, one can construct the gauge-rotated Hamiltonian

$$h \equiv \Gamma^{-1} \mathcal{H} \Gamma = -2z \frac{d^2}{dz^2} + (2\nu z^2 + 2\mu z - 1) \frac{d}{dz} - 2N\nu z + \frac{\mu}{2}, \tag{6}$$

which can be rewritten as a constant coefficient quadratic combination in terms of the \mathfrak{sl}_2 generators [2]

$$\mathcal{J}_N^+(z) = z^2 \frac{d}{dz} - N z, \quad \mathcal{J}_N^0(z) = z \frac{d}{dz} - \frac{N}{2}, \quad \mathcal{J}_N^-(z) = \frac{d}{dz}. \tag{7}$$

Explicitly,

$$h = -2 \mathcal{J}_N^0 \mathcal{J}_N^- + 2\nu \mathcal{J}_N^+ + 2\mu \mathcal{J}_N^0 - (N + 1) \mathcal{J}_N^- + \mu \left(N + \frac{1}{2} \right). \tag{8}$$

Moreover, if the parameter N takes positive integer values, then the spectral problem

$$h P(z) = E P(z), \tag{9}$$

possesses $(N + 1)$ - exact polynomial eigenfunctions P_j with $j = 0, 1, 2, \dots, N$. In this case, the $(N + 1)$ - exact solutions of the original Hamiltonian \mathcal{H} in (1) take the form

$$\psi_j(z) = P_j(z) \times \exp \left[-\frac{\nu}{4} z^2 - \frac{\mu}{2} z \right], \quad z = x^2. \quad (10)$$

All of them correspond to even-parity states, including the ground state, possessing an even number of nodes. It is worth mentioning that odd-parity exact solutions of $\mathcal{H}(1)$ appear for positive half-integer values of $N = \frac{1}{2}, \frac{3}{2}, \dots$. They correspond to excited states and can be associated with the existence of the same hidden \mathfrak{sl}_2 algebra. However, the basic object we will use later in first-order supersymmetric quantum mechanics is precisely the ground state solution. Therefore, the present study does not consider such odd-parity exact states of $\mathcal{H}(1)$ except in the part concerning the confluent second-order SUSY transformation.

3.1. Exact even-parity polynomial solutions: cases $N = 0, 1, 2, 3$

This section reviews explicit results for the lowest positive integer values of N . They are well-known in the literature and will be used later in the present consideration. For convenience, we take $\nu = 1$ and $\mu = 1$. The QES sextic potential reads

$$V^{\text{qes}}(x, N) = x^6 + 2x^4 - 2(2N + 1)x^2, \quad (11)$$

see figure 1. The corresponding exact solutions of the Schrödinger equation are the following:

- At $N = 0$, one exact solution occurs only. It is given by the nodeless function:

$$\psi_0^{(N=0)} = \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right], \quad (12)$$

with energy

$$E_0^{(N=0)} = \frac{1}{2}. \quad (13)$$

It corresponds to the exact ground state of the system. Accordingly, the polynomial factor $P_0^{(N=0)} \equiv 1$ is an eigenfunction of the \mathfrak{sl}_2 Lie-algebraic operator $h(8)$.

- At $N = 1$, two exact solutions appear:

$$\psi_0^{(N=1)} = (2x^2 + \sqrt{3} + 1) \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right], \quad (14)$$

with energy

$$E_0^{(N=1)} = \frac{3}{2} - \sqrt{3}, \quad (15)$$

and

$$\psi_2^{(N=1)} = (2x^2 - \sqrt{3} + 1) \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right], \quad (16)$$

here

$$E_2^{(N=1)} = \frac{3}{2} + \sqrt{3}. \quad (17)$$

They correspond to the ground and second excited state, respectively. The polynomial factors $P_0^{(N=1)} = 2x^2 + \sqrt{3} + 1$ and $P_2^{(N=1)} = 2x^2 - \sqrt{3} + 1$ are exact eigenfunctions of the Lie-algebraic operator $h(8)$.

- At $N = 2$, following the general theory, three exact analytical solutions occur:

$$\psi_0^{(N=2)} = (2x^4 + 6x^2 + 3) \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right], \quad (18)$$

with energy

$$E_0^{(N=2)} = -\frac{3}{2}. \quad (19)$$

Next,

$$\psi_2^{(N=2)} = (2x^4 + 2\sqrt{2}x^2 - \sqrt{2} - 1) \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right], \quad (20)$$

where

$$E_2^{(N=2)} = -2\sqrt{2} + \frac{9}{2}, \tag{21}$$

and

$$\psi_4^{(N=2)} = (2x^4 - 2\sqrt{2}x^2 + \sqrt{2} - 1)\exp\left[-\frac{1}{4}x^4 - \frac{1}{2}x^2\right], \tag{22}$$

with eigenvalue

$$E_4^{(N=2)} = 2\sqrt{2} + \frac{9}{2}. \tag{23}$$

They correspond to the ground, second, and fourth excited states. Thus, the corresponding factors $P_0^{(N=2)} = 2x^4 + 6x^2 + 3$, $P_2^{(N=2)} = 2x^4 + 2\sqrt{2}x^2 - \sqrt{2} - 1$ and $P_4^{(N=2)} = 2x^4 - 2\sqrt{2}x^2 + \sqrt{2} - 1$ are also eigenpolynomials of the Lie-algebraic operator h (8).

- At $N = 3$, four exact eigenfunctions exist:

$$\begin{aligned} \psi_k^{(N=3)} = & (384x^6 + 96(13 - 2E_k)x^4 + 12(-3 + 4(-11 + E_k)E_k)x^2 \\ & - 1263 - 22E_k + 108E_k^2 - 8E_k^3) \exp\left[-\frac{1}{4}x^4 - \frac{1}{2}x^2\right], \end{aligned} \tag{24}$$

$k = 1, 2, 3, 4$, being E_k the k th-root of the quartic equation

$$16E^4 - 224E^3 + 56E^2 + 3560E - 1191 = 0, \tag{25}$$

respectively.

3.2. QES sextic potential with energy reflection symmetry: examples with $N = 0, 1$

As previously mentioned in the introduction, at $\mu = 0$, the Hamiltonian (2) with potential $V^{\text{qes}}(x)$ as in (3) possesses the energy reflection (ER) symmetry $E \leftrightarrow -E$ [4]. For the potentials with ER symmetry, we study the two-parametric sextic potential $V_{ER}^{\text{qes}}(x)$ [26]:

$$V_{ER}^{\text{qes}}(x) = \nu^2 x^6 - \nu(4N + 2\kappa + 3)x^2, \tag{26}$$

where $\kappa = 0, 1$ indicates the parity of the algebraic solutions. The cases $\kappa = 0$ and $\kappa = 1$ are sometimes referred to as *even* and *odd* sextic QES potentials, respectively. The case $\kappa = 0$ corresponds to the potential $V^{\text{qes}}(x)$ in (3) at $\mu = 0$.

Let us revisit in detail concrete solutions using the most straightforward examples, namely $N = 0$ and $N = 1$. They will be exploited later in the confluent SUSY algorithm, a degenerate case of the second-order supersymmetric quantum mechanics:

- At $N = 0$, the potential, the single eigenfunction, and the corresponding eigenvalue are

$$V_{ER}^{\text{qes}}(x) = \nu^2 x^6 - \nu(2\kappa + 3)x^2, \quad \psi_{ER}^{(N=0)}(x) = x^\kappa \exp\left(-\frac{\nu}{4}x^4\right), \quad E_{ER}^{(N=0)} = 0, \tag{27}$$

where $\nu > 0$ and $E_{ER}^{(N=0)}$ is the energy of the ground ($\kappa = 0$) and first-excited ($\kappa = 1$) state $\psi_{ER}^{(N=0)}$, respectively.

- If $N = 1$, there are two exact analytical solutions for each value of κ , the potential, the analytic eigenfunctions, and its energies are:

$$\begin{aligned} V_{ER}^{\text{qes}}(x) = & \nu^2 x^6 - \nu(2\kappa + 7)x^2, \\ \psi_{ER,\pm}^{(N=1)}(x) = & x^k(2\nu x^2 \mp \sqrt{2\nu(1 + 2\kappa)}) \exp\left(-\frac{\nu}{4}x^4\right), \\ E_{ER,\pm}^{(N=1)} = & \pm\sqrt{2\nu(1 + 2\kappa)}. \end{aligned} \tag{28}$$

When $\kappa = 0$, the previous expressions correspond to the ground (−) and second excited (+) states whereas at $\kappa = 1$ they describe the first (−) and third excited (+) state, respectively.

4. Exact- and WKB-numerical solutions, case $N = 0$

For any value of N , by means of the user-friendly LagrangeMesh Mathematica Package [27] (LMMP), one can easily determine the *exact*-numerical eigenfunctions and eigenvalues. It is worth mentioning that the underlying Lagrange Mesh Method is an approximate variational method simplified by a Gauss quadrature associated with a certain mesh; for further details, see [27]. The high efficiency and direct control of the involved accuracy in arithmetic manipulations (and final results) allow us to obtain highly accurate energies using a personal laptop.

Table 1. Case $N = 0$: exact numerical solutions E_{exact} for the sextic potential $\frac{1}{2}V^{\text{qes}}(x, N = 0)$ defined by (11). Both the energy and the corresponding expectation value $\langle x^2 \rangle$ were obtained using the LagrangeMesh Mathematica Package [27]. Results are displayed in atomic units. For comparison, the WKB calculations E_{WKB} are presented as well. Here $\Delta E \equiv \frac{E_{\text{WKB}} - E_{\text{exact}}}{E_{\text{exact}}}$.

n	E_{exact}	E_{WKB}	$\Delta E[\%]$	$\langle x^2 \rangle$
0	0.500 000 000 000 000 000 00	0.563 698 94	0.127	0.2896023863
1	2.186 500 529 572 814 979 82	2.498 753 13	0.142	0.6490041219
2	4.871 816 665 105 781 894 19	5.223 189 89	0.072	0.8028757103
3	8.130 953 558 229 553 230 57	8.524 495 56	0.048	0.9587575624
4	11.874 846 994 114 710 710	12.303 163 1	0.036	1.0946098649
5	16.039 784 895 786 315 299	16.498 116 9	0.028	1.2170135056
6	20.581 916 287 588 558 795	21.066 969 6	0.023	1.3293330894
7	25.468 762 870 933 097 825	25.978 093 0	0.019	1.4337122075
8	30.675 013 729 847 648 104	31.206 722 1	0.017	1.5316298628
9	36.180 224 949 476 151 095	36.732 785 7	0.015	1.6241563887

Table 2. Sextic potential $\frac{1}{2}V^{\text{qes}}(x, N)$ defined by (11): the energy $E_n = E_n(N)$ in atomic units, as a function of N , for the three lowest states $n = 0, 1, 2$. As for the critical value (see text) we found $N_c \approx 0.732\ 953\ 126\ 152\ 130\ 43$. The results were obtained using the LagrangeMesh Mathematica Package [27]. For N a non-negative half (integer) number, finite exact analytical odd-parity (even-parity) eigenfunctions occur, respectively. Here [*] marks the exact analytical solutions. The present numerical results reproduce, with not less than 20 significant digits, the exact analytical ones.

N	$E_{0,\text{exact}}$	$E_{1,\text{exact}}$	$E_{2,\text{exact}}$
$-\frac{1}{4}$	0.989 580 605 436 050 838 998	3.360 990 709 529 042 484 024	6.413 007 989 851 041 421 032
$-\frac{3}{4}$	0.881 159 828 282 351 698 813	3.088 336 298 693 491 389 507	6.040 643 954 210 947 777 719
$-\frac{1}{2}$	0.764 532 033 014 503 629 516	2.802 626 174 338 946 321 036	5.659 402 903 419 812 200 032
$-\frac{1}{4}$	0.638 138 724 545 526 132 477	2.502 513 714 945 104 658 141	5.269 586 956 813 807 724 855
$-\frac{1}{8}$	0.570 681 019 914 247 060 253	2.346 594 455 340 202 692 157	5.071 641 780 887 543 784 212
0	0.500 000 000 000 000 000 00*	2.186 500 529 572 814 979 823	4.871 816 665 105 781 894 194
$\frac{1}{8}$	0.425 761 495 459 989 697 814	2.022 020 182 796 715 324 855	4.670 258 698 594 960 339 359
$\frac{1}{4}$	0.347 587 607 715 820 659 135	1.852 931 987 632 204 372 661	4.467 151 413 879 439 497 334
$\frac{1}{2}$	0.177 671 681 805 842 046 340	1.500 000 000 000 000 000 00*	4.057 235 461 202 041 394 422
N_c	9.2×10^{-12}	1.151 993 714 577 925 631 588	3.672 629 779 631 384 060 651
$\frac{3}{4}$	-0.0138436436130964772246	1.125 756 241 203 857 723 531	3.644 453 673 510 100 233 335
1	-0.2320508075688772935275*	0.728 139 096 695 212 363 575 6	3.232 050 807 568 877 293 527*
2	-1.5000000000000000000000*	-1.1383762435615163784847	1.671 572 875 253 809 902 397*
3	-3.6166170356875860239609*	-3.5323497843209945952738	0.335 095 120 779 029 553 927*

For instance, an excessive value with 30 correct significant digits can be achieved in short CPU times. A similar situation occurs for the corresponding wave functions. Therefore, in practice, we will refer to these numerical results as exact solutions, dropping the term numerical.

Just for $N = 0$, we present in table 1 the energies E_n and the expectation value $\langle x^2 \rangle$ for the lowest ten bound states $n = 0, 1, 2, \dots, 9$, explicitly. The calculations used a Hermite mesh with 800 mesh points for the energy. A comparison with WKB results is indicated as well.

In table 2, the energy $E = E(N)$ as a function of the cohomology parameter N is depicted for the three lowest states $n = 0, 1, 2$. It is worth mentioning that, in the classification of ‘quasi-exactly solvable’ Lie algebras of first-order differential operators, the parameter N can be assigned a Lie algebra cohomological interpretation [28]. If $N > -\frac{1}{2}$, the potential $V^{\text{qes}}(x, N)$ (11) has a maximum located at $x = 0$, and it corresponds to a zero energy value. It implies that for negative energies $E < 0$, the dynamics takes place in a classically forbidden region. At fixed n , the energy $E_n(N)$ is a smooth decreasing function of the parameter N . In particular, there exists a critical value $N = N_c$ such that the ground state energy $E_0(N_c)$ of the system vanishes. Accordingly, for $N > N_c$, the appearance of tunneling effects (exponentially small instanton-like terms) are expected. Interestingly, if N is a positive (half) integer number, the corresponding instanton-like terms are absent in the exact analytical solutions! This puzzle was resolved in [24] by including complex saddles. It is an interesting open question whether this is valid for non-positive (half)integer values of N .

5. QES potentials via supersymmetric quantum mechanics

5.1. First-order supersymmetry

We start from a certain Schrodinger operator $\mathcal{H}_0 = -\frac{1}{2} \frac{d^2}{dx^2} + V_0(x)$, where some of its eigenfunctions $\psi_n(x)$ and eigenenergies E_n are known explicitly. Then we suppose the existence of a first-order differential operator A_1^+ such that it intertwines \mathcal{H}_0 with a new Hamiltonian $\mathcal{H}_1 = -\frac{1}{2} \frac{d^2}{dx^2} + V_1(x)$ in the following way

$$\mathcal{H}_1 A_1^+ = A_1^+ \mathcal{H}_0, \tag{29}$$

where

$$A_1^+ = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \frac{u'(x)}{u(x)} \right). \tag{30}$$

In the last expression, $u(x)$ is called *seed solution*, and it solves the equation $\mathcal{H}_0 u = \epsilon u$, where ϵ is a constant known as *factorization energy*. We use the notation $u'(x) = du(x)/dx$. The seed solution u is not necessarily a physical solution ψ of \mathcal{H}_0 . In the literature $\alpha(x) \equiv u'(x)/u(x) = (\ln u(x))'$ is called the *superpotential*. From the intertwining relation (29) it follows the expression

$$V_1 = V_0 - (\ln u)''. \tag{31}$$

It is said that V_1 is the *SUSY partner* of V_0 . Moreover, from the equation above, the importance of avoiding zeroes of u is clear, so the SUSY partner potential V_1 is regular in the same domain of V_0 . This imposes the condition $\epsilon \leq E_0$.

If we introduce the adjoint operator of A_1^+ ,

$$A_1 = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + \frac{u'(x)}{u(x)} \right), \tag{32}$$

we can show by direct substitution that the operators A_1 and A_1^+ factorize the Hamiltonians as $\mathcal{H}_0 = A_1 A_1^+ + \epsilon$ and $\mathcal{H}_1 = A_1^+ A_1 + \epsilon$.

Also, from the intertwining relation and the factorization of \mathcal{H}_0 , it can be shown that if ψ_k is an eigenfunction of \mathcal{H}_0 with eigenvalue E_k then

$$\phi_k = \frac{1}{\sqrt{E_k - \epsilon}} A_1^+ \psi_k, \tag{33}$$

is an eigenfunction of \mathcal{H}_1 with the same eigenvalue E_k . The factor $1/\sqrt{E_k - \epsilon}$ comes from the normalization condition.

Remark. The subindex k in ψ_k indicates the energy level, i.e. ψ_k is the k -th excited state of \mathcal{H}_0 . In contrast, in the transformed wavefunction ϕ_k of \mathcal{H}_1 , it is not necessarily the k -th excited state, as we will show below.

From the factorization of \mathcal{H}_1 , we can see that the function annihilated by A_1 is also a solution of the eigenvalue equation of \mathcal{H}_1 with eigenvalue ϵ . This function is known as the *missing state* and its expression is

$$\phi_\epsilon \propto \frac{1}{u}. \tag{34}$$

If ϕ_ϵ fulfills the boundary conditions, ϵ belongs to the spectrum of \mathcal{H}_1 , otherwise it does not. In general, if $\epsilon_0 = E_0$ and $u(x) = \psi_0$ then $\text{Sp}(\mathcal{H}_1)$ differs from $\text{Sp}(\mathcal{H}_0)$ only on the ground state energy.

Below, we present the supersymmetric partner potential $V_1(x, N)$ corresponding to $V_0 = \frac{1}{2} V^{\text{qes}}(x, N)$ for the lowest cases $N = 0, 1, 2, 3$.

5.2. 1-SUSY partner potential V_1 with $N = 0$

We take $V_0 = \frac{1}{2} V^{\text{qes}}(x, N = 0)$. In this case, we solely know the exact ground state solution $\psi_0^{(N=0)}$ with energy $E_0 = 1/2$. No exact analytical solutions for the excited states of \mathcal{H}_0 are known. Then, we can only choose $u = \psi_0^{(N=0)}$. Using (31), we immediately obtain the SUSY partner potential

$$V_1(x, N = 0) = \frac{x^6}{2} + x^4 + 2x^2 + 1, \tag{35}$$

which is in complete agreement with the general equation (31) in [6], see Figure 2.

Up to an additive constant, V_1 (35) coincides with the sextic potential $\frac{1}{2} V^{\text{qes}}(x, N = -3/2)$, see (11). Thus, just like V_0 , it is an analytic function with no poles on the real (and complex) domain. Remarkably, the SUSY partner Hamiltonian \mathcal{H}_1 with potential (35) still possesses a hidden \mathfrak{sl}_2 Lie algebraic structure but with a negative value of parameter N . The trivial SUSY solution of $\mathcal{H}_1 \phi = E \phi$, namely, $\phi = 1/u$ is non-square integrable.

5.3. 1-SUSY partner potential V_1 with $N = 1$

Here $V_0 = \frac{1}{2}V^{\text{qes}}(x, N = 1)$, for which we know two exact analytical solutions, namely $\psi_0^{(N=1)}$ and $\psi_2^{(N=1)}$. Substituting $u = \psi_0$ in (31) we obtain

$$\begin{aligned} V_1(x, N = 1) &= \frac{x^6}{2} + x^4 + 1 - \frac{4(-2x^2 + \sqrt{3} + 1)}{[P_0^{(N=1)}]^2} \\ &= \frac{1}{2}V^{\text{qes}}(x, N = -1/2) + 1 - \frac{4(-2x^2 + \sqrt{3} + 1)}{[P_0^{(N=1)}]^2}, \end{aligned} \quad (36)$$

where $P_0^{(N=1)} = 2x^2 + \sqrt{3} + 1$ vanishes only in the complex domain. The SUSY transformation retrieves a rational extension of V^{qes} . For the above potential V_1 , we know only a single exact solution (the first excited state)

$$\phi_2 = A_1^+ \psi_2 = \frac{x}{P_0^{(N=1)}} \exp\left(-\frac{1}{4}x^4 - \frac{1}{2}x^2\right), \quad (37)$$

with energy $E_2^{(N=1)} = \frac{3}{2} + \sqrt{3}$. It is very common that the SUSY transformation changes the parameters of a potential and adds a finite term. In this case, such a finite part is a rational function in the variable x^2 , with complex poles given by the complex zeros of $P_0^{(N=1)}$. Figure 3 Left shows the potential V_0 (blue) and its SUSY partner V_1 (red); and Figure 3 Right the analytic eigenfunction ϕ_2 , corresponding to the first excited state of \mathcal{H}_1 , note it has a single node at $x=0$.

Now, towards the construction of the algebraic operator which governs the polynomial part in (37), let us introduce the gauge factor

$$\Gamma_1 = \frac{1}{P_0^{(N=1)}} \exp\left[-\frac{1}{4}x^4 - \frac{1}{2}x^2\right], \quad (38)$$

thus, $\phi_2 = x \Gamma_1$ and the polynomial factor of ϕ_2 is simply x . Clearly, in the x -variable the gauge rotated Hamiltonian $\Gamma_1^{-1} \mathcal{H}_1 \Gamma_1$ will contain non-polynomial coefficients. To obtain an algebraic differential operator with polynomial coefficients we construct the gauge-rotated Hamiltonian

$$\begin{aligned} h^{(susy)}|_{N=1} &\equiv 2 P_0^{(N=1)} \Gamma_1^{-1} (\mathcal{H}_1 - \lambda) \Gamma_1 \\ &= -(2x^2 + \sqrt{3} + 1) \frac{d^2}{dx^2} + 2x[2x^4 + (\sqrt{3} + 3)x^2 + \sqrt{3} + 5] \frac{d}{dx} \\ &\quad - [4x^4 - 2(\sqrt{3} - 2\lambda)x^2 - 3\sqrt{3} + 1 + 2(\sqrt{3} + 1)\lambda], \end{aligned} \quad (39)$$

where λ is a real constant. That way, the spectral problem $\mathcal{H}_1 \phi = E \phi$ is equivalent to the zero-mode equation $h^{(susy)} p(x) = 0$ being $p = p(x)$ a function to be determined. In this case, $\phi_2(x) = p(x)\Gamma_1$ is the solution of the Hamiltonian \mathcal{H}_1 .

By construction, at $\lambda = E_2^{(N=1)} = \frac{3}{2} + \sqrt{3}$, the polynomial $p_1 = x$ belongs to the kernel of $h^{(susy)}|_{N=1}$, namely $h^{(susy)}|_{N=1} p_1 = 0$. Nevertheless, in the x -variable this operator can not be rewritten as a constant coefficient quadratic combination in the \mathfrak{sl}_2 generators (7).

5.4. 1-SUSY partner potential V_1 with $N = 2$

Here $V_0 = \frac{1}{2}V^{\text{qes}}(x, N = 2)$, for which we know three exact analytical solutions, $\psi_0^{(N=2)}$, $\psi_2^{(N=2)}$ and $\psi_4^{(N=2)}$. Substituting $u = \psi_0$ in (31) we obtain

$$\begin{aligned} V_1(x, N = 2) &= \frac{x^6}{2} + x^4 - 2x^2 + 1 + \frac{4(2x^2 - 3)}{P_0^{(N=2)}} + \frac{48x^2}{[P_0^{(N=2)}]^2} \\ &= \frac{1}{2}V^{\text{qes}}(x, N = 1/2) + 1 + \frac{4(2x^2 - 3)}{P_0^{(N=2)}} + \frac{48x^2}{[P_0^{(N=2)}]^2}, \end{aligned} \quad (40)$$

here $P_0^{(N=2)} = 2x^4 + 6x^2 + 3$. For the above potential V_1 we know two exact solutions: a first excited state

$$\phi_2 = \frac{((6\sqrt{2} - 4)x^5 + (8\sqrt{2} + 4)x^3 + 3(\sqrt{2} + 4)x)}{P_0^{(N=2)}} \exp\left(-\frac{1}{4}x^4 - \frac{1}{2}x^2\right), \quad (41)$$

with energy $E_2^{(N=2)} = \frac{9}{2} - 2\sqrt{2}$, and a third excited state

$$\phi_4 = \frac{((6\sqrt{2} + 4)x^5 + (8\sqrt{2} - 4)x^3 + 3(\sqrt{2} - 4)x)}{P_0^{(N=2)}} \exp\left(-\frac{1}{4}x^4 - \frac{1}{2}x^2\right), \quad (42)$$

with energy $E_4^{(N=2)} = \frac{9}{2} + 2\sqrt{2}$, respectively. Figure 4 shows, on the left, the potential V_1 (red) and for reference V_0 (blue); and on the right, the first and third excited states, ϕ_2 (blue) and ϕ_4 (red).

Now, let us introduce the gauge factor

$$\Gamma_2 = \frac{1}{P_0^{(N=2)}} \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right]. \tag{43}$$

Again, in the x -variable the gauge rotated Hamiltonian $\Gamma_2^{-1} \mathcal{H}_1 \Gamma_2$ will contain non-polynomial coefficients. In order to obtain an algebraic differential operator with polynomial coefficients we build the gauge-rotated Hamiltonian

$$\begin{aligned} h^{(susy)}|_{N=2} &\equiv 2(2x^4 + 6x^2 + 3)\Gamma_2^{-2}(\mathcal{H}_1 - \lambda)\Gamma_2 \\ &= -(2x^4 + 6x^2 + 3)\frac{d^2}{dx^2} + 2x[2x^6 + 8x^4 + 17x^2 + 15]\frac{d}{dx} \\ &\quad - [20x^6 + (4\lambda + 46)x^4 + 12(\lambda + 3)x^2 + 3 + 6\lambda], \end{aligned} \tag{44}$$

where λ is a real constant. That way, the spectral problem $\mathcal{H}_1 \phi = E \phi$ is equivalent to the zero-mode equation $h^{(susy)} p = 0$.

By construction, at $\lambda = \frac{9}{2} \pm 2\sqrt{2}$, the kernel of $h^{(susy)}|_{N=2}$ admits two fifth order polynomial solutions $p_{5,\pm}$, i.e. they obey $h^{(susy)}|_{N=2} p_{5,\pm} = 0$. Nevertheless, in the x -variable, this operator can not be rewritten as a constant coefficient quadratic combination in the \mathfrak{sl}_2 generators (7).

5.5. 1-SUSY partner potentials V_1 with arbitrary integer $N > 0$

Here $V_0 = \frac{1}{2} V^{qes}(x, N)$, for which we know $(N + 1)$ exact solutions, $\psi_0, \psi_2, \dots, \psi_N$. Substituting $u = \psi_0$ in (31) we arrive to the following expression

$$V_1(x, N) = \frac{1}{2} V^{qes}(x, N - 3/2) + 1 + \frac{Q_{N-1}(x^2)}{P_0^{(N)}} + \frac{R_{[N/2]}(x^2)}{[P_0^{(N)}]^2}, \tag{45}$$

here $P_0^{(N)} = P_0^{(N)}(x^2)$ is the $2N -$ order (in $x -$ variable) exact ground state polynomial solution of h (6), whereas Q_{N-1} and $R_{[N/2]}$ are polynomial functions in x^2 of order $(N - 1)$ and $[N/2]$, respectively³. For the Hamiltonian \mathcal{H}_1 with the above potential V_1 (45), the N known exact solutions are given by (33). In (45), the part containing no complex poles is given by the sextic potential $\frac{1}{2} V^{qes}(x, N)$ with the replacement $N \rightarrow N - \frac{3}{2}$.

Remark. At large distances $|x| \rightarrow \infty$, the last two rational terms in (45) vanish. The dominant term behaves as $\frac{1}{x^2}$. Thus, in this limit the asymptotic behaviour $\phi \sim \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right]$ of the solutions for V_1 is the same as that occurring for V^{qes} .

Using the gauge factor

$$\Gamma_N = \frac{1}{P_0^{(N)}} \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right], \tag{46}$$

we construct the gauge-rotated Hamiltonian

$$\begin{aligned} h^{(susy)}|_N &\equiv 2 P_0^{(N)} \Gamma_N^{-2} (\mathcal{H}_1 - \lambda) \Gamma_N \\ &= -P_0^{(N)} \frac{d^2}{dx^2} + 2x Y_{N+1} \frac{d}{dx} - Z_{N+1}, \end{aligned} \tag{47}$$

where λ is a real constant and Y_{N+1}, Z_{N+1} are polynomial functions in x^2 of the same order $(N + 1)$. Hence, the spectral problem $\mathcal{H}_1 \phi = E \phi$ is equivalent to the zero-mode equation $h^{(susy)}|_N p = 0$.

By construction, the kernel of $h^{(susy)}|_N$ admits N polynomial solutions p_{4N-3} of order $(4N - 3)$, i.e. they obey $h^{(susy)}|_N p_{4N-3} = 0$. Nevertheless, in the x -variable, this operator can not be rewritten as a constant coefficient quadratic combination in the \mathfrak{sl}_2 generators (7).

Moreover, we can recover an intertwining relation for the Hamiltonian h given in (6). From the intertwining relation (29) we can define the SUSY partner of h as $h_1 = \Gamma_N^{-1} h \Gamma_N$ and the intertwining operator as $\mathcal{A}_1^\dagger = \Gamma_N^{-1} \mathcal{A}_1^\dagger \Gamma$, then

$$h_1 \mathcal{A}_1^\dagger = \mathcal{A}_1^\dagger h. \tag{48}$$

Also, by construction, if $P(z)$ solves $hP(z) = EP(z)$ then $\mathcal{P}(z) = \mathcal{A}_1^\dagger P(z)$ solves $h_1 \mathcal{P}(z) = E \mathcal{P}(z)$.

³ $[a]$ denotes the integer part of a .

6. Confluent 2-SUSY generating quasi-exactly solvable potentials

6.1. Second-order supersymmetry: confluent case

Let us present this second-order SUSY transformation as an iteration of two first-order transformations; see more details in [29–31]. Based on the results of the previous subsection 5.1, from \mathcal{H}_1 we will construct a Hamiltonian $\mathcal{H}_2 = -\frac{1}{2} \frac{d^2}{dx^2} + V_2(x)$ using a second intertwining operator A_2^+ , namely

$$\mathcal{H}_2 A_2^+ = A_2^+ \mathcal{H}_1, \quad A_2^+ = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \frac{v'(x)}{v(x)} \right), \quad (49)$$

where the seed function $v(x)$ solves the spectral problem $\mathcal{H}_1 v = \epsilon v$. Note that we have used the same factorization energy ϵ of the previous SUSY step; this is the characteristic feature of the confluent SUSY transformation. The SUSY partner potential becomes

$$V_2 = V_1 - (\ln v)'' . \quad (50)$$

A simple choice of v comes from (33), $v = 1/u$; however, this selection leads to $V_2 = V_0$. From the reduction-of-order formula, the general choice is given by

$$v(x) = \frac{1}{u(x)} \left(\omega_0 + \int^x u^2(y) dy \right), \quad (51)$$

where ω_0 is a constant. To simplify notation, we can define

$$\omega(x) \equiv \omega_0 + \int^x u^2 dy. \quad (52)$$

Accordingly, the second-order confluent SUSY partner potential of V_0 is

$$V_2(x) = V_0(x) - (\ln \omega(x))'' . \quad (53)$$

Note that there is an intertwining relation between \mathcal{H}_0 and \mathcal{H}_2 : $\mathcal{H}_2 B^+ = B^+ \mathcal{H}_0$, where $B^+ = A_2^+ A_1^+$. Moreover, the operator B^+ and its adjoint $B = (B^+)^+$ factorize as follows:

$$B^+ B = (\mathcal{H}_2 - \epsilon)^2, \quad B B^+ = (\mathcal{H}_0 - \epsilon)^2. \quad (54)$$

The missing state of \mathcal{H}_2 becomes

$$\phi_\epsilon \propto \frac{1}{v} = \frac{u}{\omega(x)}, \quad (55)$$

and the eigenfunctions of \mathcal{H}_0 are mapped to those of \mathcal{H}_2 as

$$\phi_n = \frac{1}{E_n - \epsilon} B^+ \psi_n, \quad E_n \neq \epsilon. \quad (56)$$

Below, we display new results in the QES literature on the confluent SUSY partners of the potentials with energy reflection symmetry presented in subsection 3.2. To simplify notation, we set $\nu = 1$.

6.2. Even parity with $N = 0$

To obtain the confluent SUSY partner of $V_0 = \frac{1}{2} V_{ER}^{\text{qes}}(x; N = 0, \kappa = 0) = \frac{1}{2}(x^6 - 3x^2)$, besides selecting the seed function which in this case is $u = \psi_{ER}^{N=0}(x; \kappa = 0)$ (27), it is important to calculate

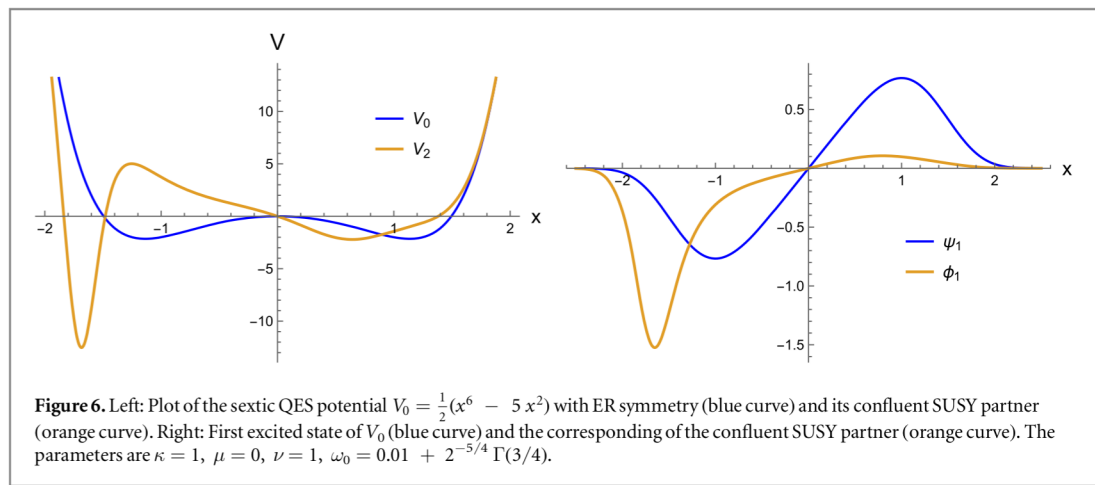
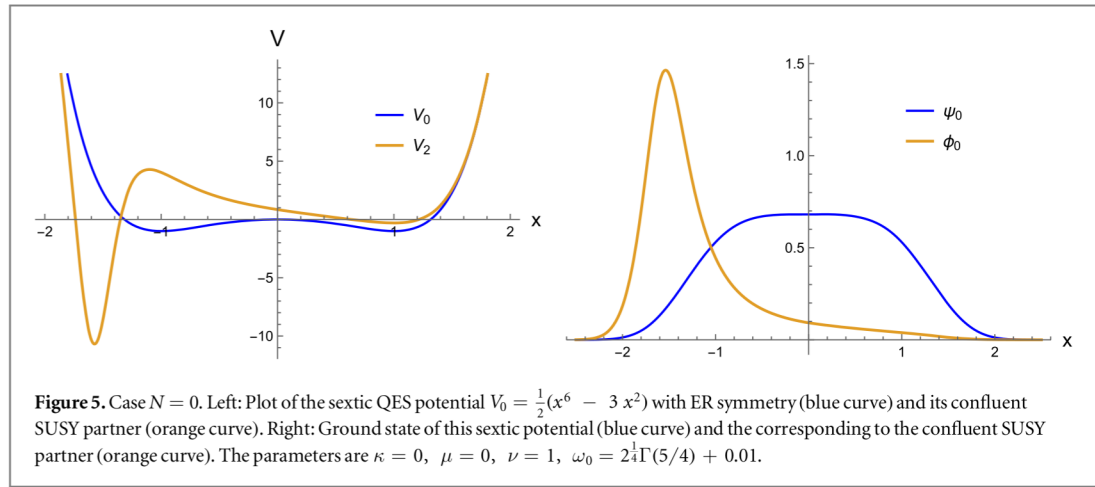
$$\omega(x) = \omega_0 + \int_0^x u^2(y) dy = \omega_0 + 2^{\frac{1}{4}} \Gamma\left(\frac{5}{4}\right) - \frac{1}{4} x E_{\frac{3}{4}}\left(\frac{x^4}{2}\right), \quad (57)$$

where $\Gamma(z)$ is the Gamma function, and $E_n(z)$ is the exponential integral function $E_n(z) = \int_1^\infty e^{-zt}/t^n dt$. We fixed the lower limit of the integral $x_0 = 0$ to use the parity properties of u^2 . It is necessary to avoid zeros of $\omega(x)$, for this reason $\omega_0 \in (-\infty, -2^{\frac{1}{4}} \Gamma(5/4)) \cup (2^{\frac{1}{4}} \Gamma(5/4), \infty)$ to guarantee the regularity of the potential $V_2 = V_0 - (\ln \omega(x))''$. There is a single analytic eigenfunction (55) of \mathcal{H}_2

$$\phi_0 = \frac{\exp\left(-\frac{1}{4}x^4\right)}{\omega_0 + 2^{\frac{1}{4}} \Gamma\left(\frac{5}{4}\right) - \frac{1}{4} x E_{\frac{3}{4}}\left(\frac{x^4}{2}\right)}, \quad (58)$$

with eigenvalue $\epsilon = 0$.

The corresponding confluent SUSY partner potential $V_2(x; \kappa = 0, N = 0)$ and its exact ground state function (58) are shown in figure 5.



6.3. Odd parity with $N = 0$

For the potential $V_0 = \frac{1}{2}V_{ER}^{qes}(x; N = 0, \kappa = 1) = \frac{1}{2}(x^6 - 5x^2)$, we only have the exact first-excited state. Consequently, using a first-order SUSY transformation will be impossible without adding a singularity to the potential. The confluent-SUSY transformation allows us to create an isospectral Hamiltonian with non-singular potential. In this case $u = \psi_{ER}^{N=0}(x; \kappa = 1) = x \exp(-x^4/4)$, see (27), then:

$$\omega(x) = \omega_0 + \int_0^x u^2(y) dy = \omega_0 + \frac{1}{4} \left(2^{3/4} \Gamma\left(\frac{3}{4}\right) - x^3 E_4\left(\frac{x^4}{2}\right) \right),$$

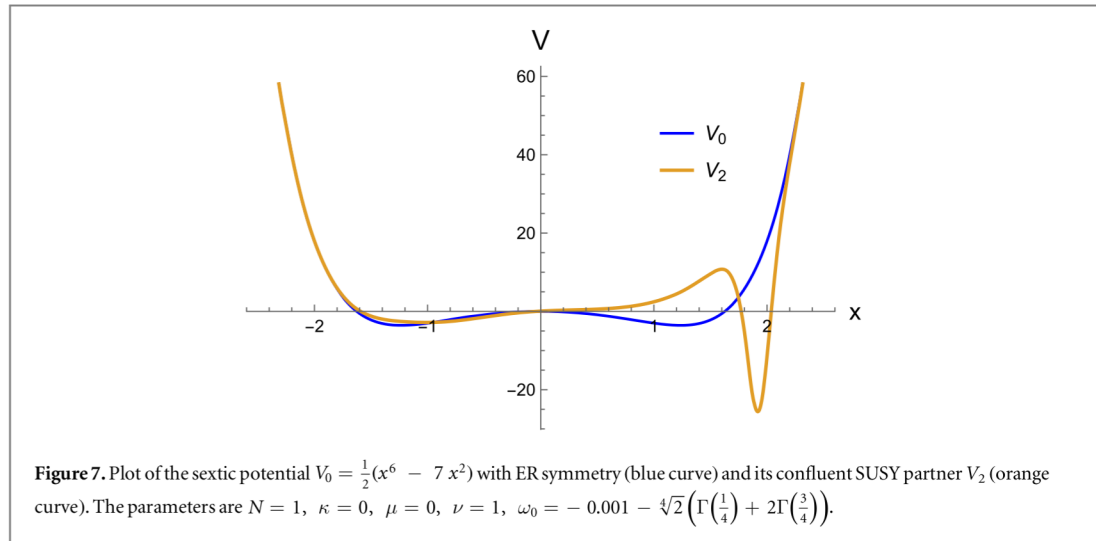
$$\phi_0 = \frac{x \exp(-x^4/4)}{\omega_0 + \frac{1}{4} \left(2^{3/4} \Gamma\left(\frac{3}{4}\right) - x^3 E_4\left(\frac{x^4}{2}\right) \right)}. \tag{59}$$

To avoid singularities on the real line in the potential $V_2 = V_0 - (\ln \omega(x))''$, the function $\omega(x)$ must be nodeless, thus $\omega_0 = (-\infty, -2^{-5/4}\Gamma(3/4)) \cup (2^{-5/4}\Gamma(3/4), \infty)$. The corresponding confluent SUSY partner potential $V_2(x; \kappa = 1, N = 0)$ and its exact fist-excited state function (59) are shown in figure 6.

Remark. For the confluent SUSY partner potential $V_2(x; N = 0)$ the original ER symmetry remains. However, the wave functions of the ER symmetric levels are not connected by the original analytical continuation $x \rightarrow ix$.

6.4. An example with $N = 1$

Since, at $N = 1$, there are two known exact eigenfunctions of \mathcal{H}_0 , there are two different choices for the seed functions $u(x)$. For even parity $\kappa = 0$, they are the ground and the second excited states. For odd parity $\kappa = 1$, they correspond to the first and the third excited states. The confluent algorithm can be applied as in the previous cases. Firstly, we pick the seed function ($u = \psi_{ER,\pm}^{(N=1)}$, see (28)); so we can construct the function $\omega(x)$ defined as



in (52). Secondly, it is important to find the domain of ω_0 such that $\omega(x)$ is nodeless. Thirdly, it is straightforward to obtain the confluent SUSY partner V_2 (see (53)). Eventually, there will be two eigenfunctions of \mathcal{H}_2 associated with the same energies of the initial system \mathcal{H}_0 , one of them ϕ_ϵ is constructed as in (55) and the second one ϕ_n is calculated using (56). As a result, we can build four different Hamiltonians because, for each parity, we have two options for the seed function.

6.4.1. Even parity, $u = \psi_{ER,-}^{(N=1)}$

In this case $u = \psi_{ER,-}^{(N=1)}(x; \kappa = 0) = (2x^2 + \sqrt{2}) \exp\left(-\frac{1}{4}x^4\right)$, then:

$$\begin{aligned} \omega(x) &= \omega_0 - xE_4^3 \left(\frac{x^4}{2}\right) - \sqrt{2}x^3E_4^1 \left(\frac{x^4}{2}\right) - 2e^{-\frac{x^4}{2}}x - \frac{\Gamma\left(-\frac{1}{4}\right) - 2\Gamma\left(\frac{1}{4}\right)}{2^{3/4}}, \\ \phi_0 &= \frac{(2x^2 + \sqrt{2}) \exp\left(-\frac{1}{4}x^4\right)}{\omega(x)}, \quad \epsilon = -\sqrt{2}, \\ \phi_2 &= \frac{2}{\sqrt{2} + 2x^2} \exp\left(-\frac{x^4}{2}\right) \left[1 - 2x^4 - \frac{\exp\left(-\frac{x^4}{2}\right)x(2 + 4x^2(\sqrt{2} + x^2))}{\omega(x)} \right], \end{aligned} \quad (60)$$

and $E = \sqrt{2}$. To avoid singularities on the real line in the potential $V_2 = V_0 - (\ln \omega(x))''$, the function $\omega(x)$ must be nodeless, thus $\omega_0 \in \left(-\infty, -\sqrt[4]{2} \left[\Gamma\left(\frac{1}{4}\right) + 2\Gamma\left(\frac{3}{4}\right) \right] \right) \cup \left(\sqrt[4]{2} \left[\Gamma\left(\frac{1}{4}\right) + 2\Gamma\left(\frac{3}{4}\right) \right], \infty \right)$. The corresponding confluent SUSY partner potential $V_2(x; \kappa = 0, N = 1)$, their exact ground state ϕ_0 and second excited state ϕ_2 eigenfunctions (60) are shown in figures 7 and 8.

7. SUSY in the non-algebraic sector of V^{qes} : case $N = 0$

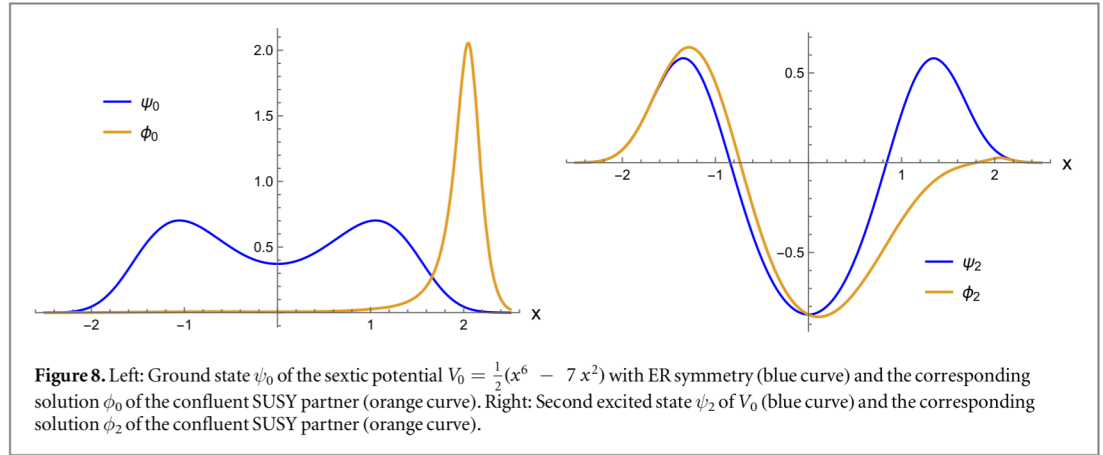
Let us consider again the case $N = 0$. For the QES sextic potential

$$V_0(x) = \frac{1}{2}V^{\text{qes}}(x, N = 0) = x^6 + 2x^4 - 2x^2,$$

only the ground state function ψ_0 (12) is known analytically. The corresponding 1-SUSY partner potential $V_1(x)$ can be found immediately. Even though no more exact analytical solutions exist other than ψ_0 , it is important to emphasize that both potentials, V_0 and V_1 , as well as their corresponding (unknown) eigenfunctions and eigenvalues, are still connected by SUSY means.

The set of excited states of V_0 can be calculated using numerical and approximate methods. In particular, for one-dimensional systems, a SUSY scheme based on a hierarchy of Hamiltonians has been introduced in previous works to compute the excited states (see [14] and references therein). However, the accuracy of the so obtained solutions is rather limited.

Therefore, it would be worth analyzing the effect of a SUSY transformation acting on the approximate solutions of V_0 . This would generate approximate solutions for V_1 . The accuracy of approximate solutions can



be easily estimated using a direct numerical method. Hence, the concrete question we aim to answer can be formulated as follows: *How does the accuracy of approximate solutions change when a SUSY transformation is applied?* This will allow us to determine the applicability of SUSY techniques in the non-algebraic sector of the QES systems.

We adopt the variational approach to compute the first excited states of $V_0(x)$ as a first step. The corresponding trial functions $\psi_{\text{trial}}(x)$ are designed on physical grounds and a criterion of simplicity. The accuracy of the obtained solutions, energies, and wave functions is estimated. Afterward, using the operator A_1^+ , we calculate approximate solutions $\phi = A_1^+ \psi_{\text{trial}}$ for V_1 and determine how the accuracy is modified.

7.1. $V_0 = \frac{1}{2}V^{\text{qes}}(x, N=0)$: approximate variational solutions

7.1.1. First excited state

For the first excited state of V_0 , we employ the trial function

$$\psi_{\text{trial}}^{(1\text{st})}(x; c_i, k) = \left(\sum_{i=0}^k c_i x^{2i+1} \right) \times \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right], \quad (61)$$

where the c_i , ($i = 0, 1, 2, \dots, k$), are $(k+1)$ -variational parameters to be determined by the minimization procedure of the energy functional. As a result of calculations, we can always put $c_0 = 1$. The function (61) possesses the following properties:

- the orthogonality condition with the exact ground state function is satisfied identically, $\langle \psi_{\text{trial}}^{(1\text{st})} | \psi_0^{\text{(exact)}} \rangle = 0$.
- it possesses a definite odd parity $\psi_{\text{trial}}(-x) = -\psi_{\text{trial}}(x)$.
- the node is correctly located at $x = 0$.
- the exponential $\sim \exp \left(-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right)$ factor captures the exact asymptotic behaviour of the state.

The corresponding energy functional

$$E_{\text{var}}[\psi_{\text{trial}}; c_i] = \frac{\langle \psi_{\text{trial}}^{(1\text{st})} | \mathcal{H} | \psi_{\text{trial}}^{(1\text{st})} \rangle}{\langle \psi_{\text{trial}}^{(1\text{st})} | \psi_{\text{trial}}^{(1\text{st})} \rangle}, \quad (62)$$

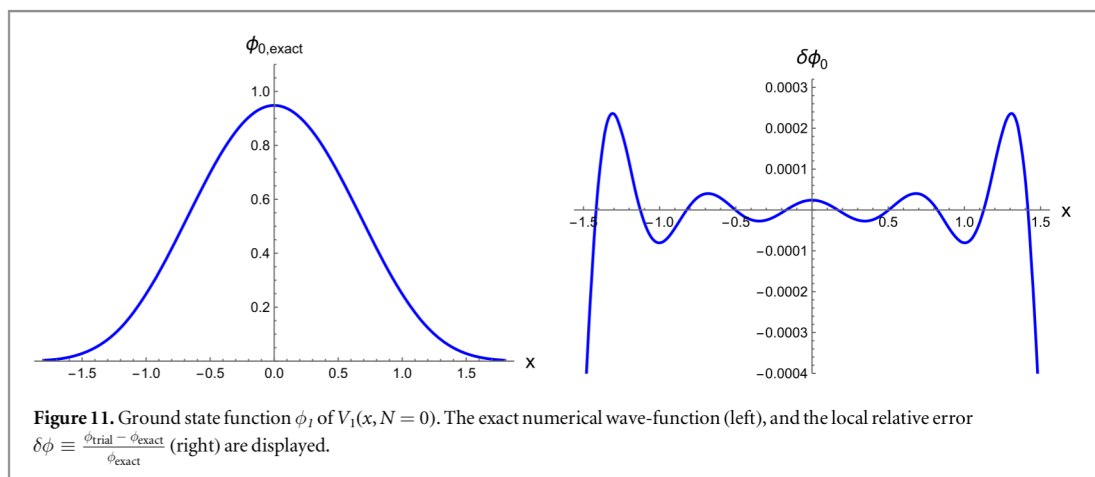
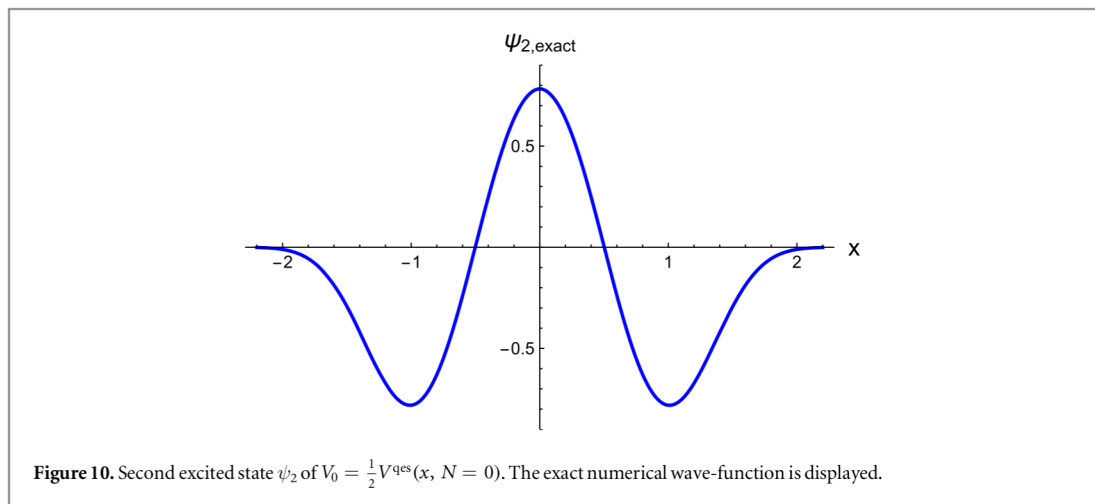
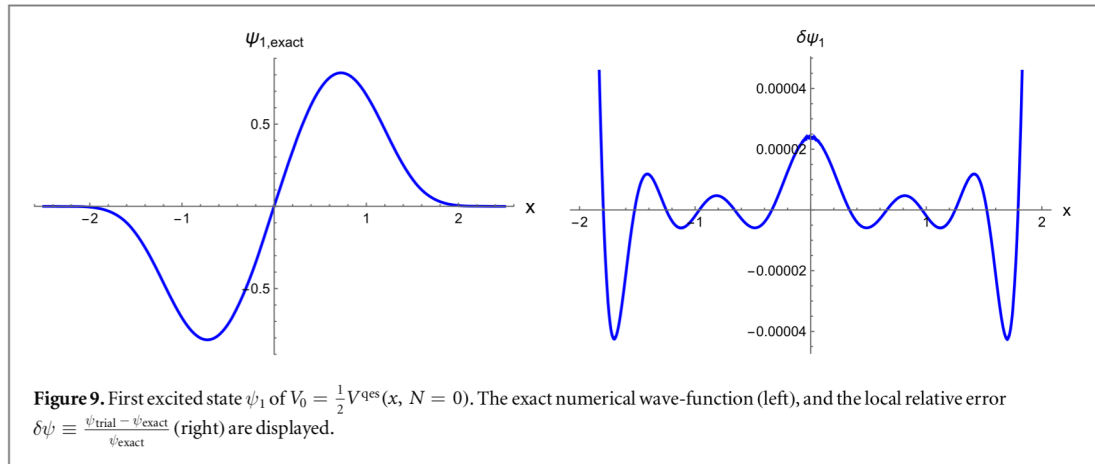
can be evaluated analytically in terms of Bessel functions. However, we do not present the corresponding (lengthy) expression explicitly.

For the lowest values of $k = 1, 2, \dots, 6$, the minimization of (62) gives the results of the energy $E_{\text{var}}^{(1\text{st})}$ displayed in table 3. Using the highly accurate LagrangeMesh Mathematica Package[27] we obtain the corresponding *exact* numerical result

$$E_{\text{exact}}^{(1\text{st})} = 2.186\,500\,529\,572\,814\,979\,822\,675 \text{ a.u.} \quad (63)$$

At $k = 6$, the optimal variational parameters are $c_0 = 1$,

$$\begin{aligned} c_1 &= -0.2284993256, & c_2 &= 0.068\,573\,912\,26, & c_3 &= -0.01921167947 \\ c_4 &= 0.0042010480, & c_5 &= -0.0005803713, & c_6 &= 0.000\,035\,814\,8 \end{aligned}$$



provide a relative error $e_r \equiv \frac{E_{\text{var}} - E_{\text{exact}}}{E_{\text{exact}}}$ of order $\approx 10^{-9}$ with respect to the above exact value $E_{\text{exact}}^{(1st)}$. The corresponding relative (local) error $\delta\psi \equiv \frac{\psi^{\text{trial}} - \psi^{\text{exact}}}{\psi^{\text{exact}}}$, of order 10^{-5} or less, in the wave-function is shown in figure 9.

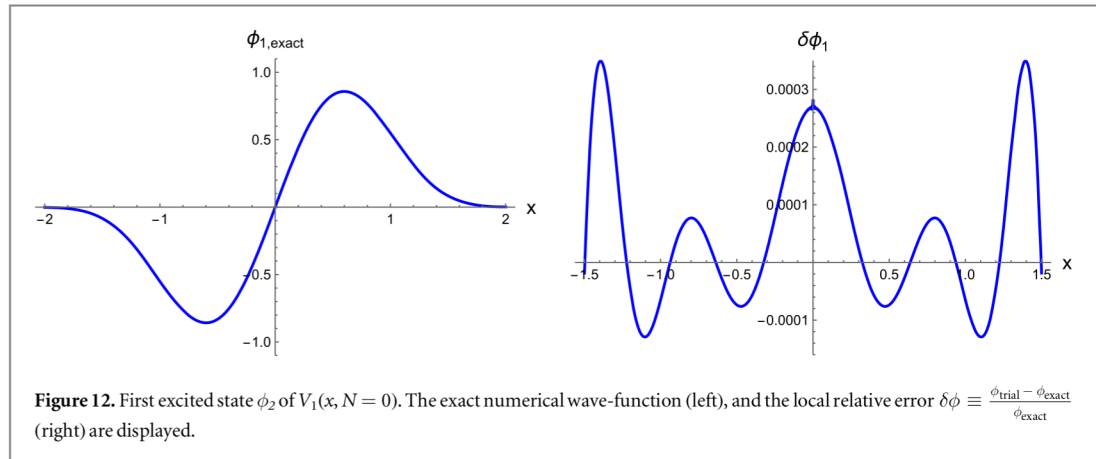


Table 3. First excited state of $V_0 = \frac{1}{2}V^{\text{qes}}(x, N = 0)$: variational energy obtained using (61) for the lowest values of $k = 1, 2, 3, 4, 5, 6$. The relative error $e_r \equiv \frac{E_{\text{var}} - E_{\text{exact}}}{E_{\text{exact}}}$ is displayed.

k	$E_{\text{var}}^{(1\text{st})}$	e_r
1	2.188 451 041	0.000 892
2	2.186 607 928	0.000 049
3	2.186 506 914	2.92×10^{-6}
4	2.186 500 932	1.84×10^{-7}
5	2.186 500 556	1.22×10^{-8}
6	2.186 500 531	8.338×10^{-10}

7.1.2. Second excited state

Similarly, for the second excited state of $V_0 = \frac{1}{2}V^{\text{qes}}(x, N = 0)$ we use the following trial function

$$\psi_{\text{trial}}^{(2\text{nd})}(x; b_i, k) = \left(\sum_{i=0}^k b_i x^{2i} \right) \times \exp \left[-\frac{1}{4}x^4 - \frac{1}{2}x^2 \right], \quad (64)$$

where the b_i , ($i = 0, 1, 2, \dots, k$), are $(k + 1)$ -variational parameters to be determined by the minimization procedure of the energy functional and the orthogonality condition. For the values of $k = 4, 5, 6, 7$, the energy $E_{\text{var}}^{(2\text{nd})}$ is presented in table 4.

At $k = 7$, the optimal variational parameters are $b_0 = 1$,

$$\begin{aligned} b_1 &= -4.3707510870, & b_2 &= 1.7193375149, & b_3 &= -0.6002796514, \\ b_4 &= 0.1742370133, & b_5 &= -0.0370162648, & b_6 &= 0.0048080288, \\ b_7 &= -0.0002756139 \end{aligned} \quad (65)$$

provide the energy $E_{\text{var}}^{(2\text{nd})} = 4.8718167005$ a.u. with a relative error $e_r \equiv \frac{E_{\text{var}} - E_{\text{exact}}}{E_{\text{exact}}}$ of order $\approx 10^{-8}$ with respect to the exact value

$$E_{\text{exact}}^{(2\text{nd})} = 4.8718166651057818941944 \text{ a.u.},$$

obtained numerically with the LagrangeMesh Mathematica Package[27], see Figure 10. The local relative accuracy of $\psi_{\text{trial}}^{(2\text{nd})}$ (64) is of order 10^{-4} or less. In particular, the two nodes of (64) are located at $x \approx \pm 0.5017$ in agreement with the exact numerical result.

7.2. Ground state and first excited state of $V_1(x)$: approximate SUSY solutions

Now, for the 1-SUSY partner potential V_1 (35) we calculate the approximate ground state solution $\phi_{1,\text{trial}}$ by simply acting the operator A_1^+ (30) on the first excited state $\psi_{\text{trial}}^{(1\text{st})}$ defined in (61). No further variational minimization is involved. Explicitly,

Table 4. Second excited state of $V_0 = \frac{1}{2}V^{\text{qes}}(x, N = 0)$: variational energy obtained using (64) for the values of $k = 4, 5, 6, 7$.

k	$E_{\text{var}}^{(2\text{nd})}$	e_r
4	4.871 903 126 3	0.000 018
5	4.871 823 001 4	1.3×10^{-6}
6	4.871 817 135 3	9.6×10^{-8}
7	4.871 816 700 5	7.3×10^{-9}

Table 5. Ground state energy of $V_1(x; N = 0)$: the energy $\epsilon_{1,\text{var}}$ is obtained calculating the expectation value of \mathcal{H}_1 on the function $\phi_{1,\text{trial}} = A_1^+ \psi_{\text{trial}}^{(1\text{st})}$. The relative error $e_r \equiv \frac{\epsilon_{\text{var}} - \epsilon_{\text{exact}}}{\epsilon_{\text{exact}}}$ is displayed as well.

k	$\epsilon_{1,\text{var}}$	e_r
1	2.204 364 851 9	0.008
2	2.188 080 928 6	0.000 72
3	2.186 634 665 9	0.000 061
4	2.186 511 781 5	5.15×10^{-6}
5	2.186 501 472 8	4.31×10^{-7}
6	2.186 500 608 9	3.63×10^{-8}

$$\phi_{1,\text{trial}} = A_1^+ \psi_{\text{trial}}^{(1\text{st})} = \left(\sum_{i=0}^k c_i (2i+1) x^{2i} \right) \times \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right], \quad (66)$$

where the values of parameters c_i correspond to the optimal results obtained for $\psi_{\text{trial}}^{(1\text{st})}$ previously. Figure 11 displays the ground state wavefunction $\phi_{1,\text{exact}}$ (left) and the relative error (right). The energy obtained using $\phi_{1,\text{trial}}(x)$ as a variational function with no-free parameters is presented in table 5.

Next, we compute the approximate first excited state of V_1 acting the operator A_1^+ (30) onto the second excited state $\psi_{\text{trial}}^{(2\text{nd})}$ defined in (64). Explicitly,

$$\phi_{2,\text{trial}} = A_1^+ \psi_{\text{trial}}^{(2\text{nd})} = \left(\sum_{i=1}^7 2 i b_i x^{2i-1} \right) \times \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right], \quad (67)$$

where the optimal values of parameters b_i are taken from (65), see Figure 12. In this case the relative error $e_r \equiv \frac{\epsilon_{\text{var}} - \epsilon_{\text{exact}}}{\epsilon_{\text{exact}}}$ is of order $\approx 10^{-7}$ with respect to the exact value $\epsilon_2 = E_{\text{exact}}^{(2\text{nd})}$. By construction, the orthogonality condition $\langle \phi_{2,\text{trial}} | \phi_{1,\text{trial}} \rangle = 0$ is fulfilled exactly.

Therefore, the above results indicate that the relative accuracy of the optimal approximate solutions is not preserved under the SUSY transformation. It generically decreases by one order of magnitude (at most).

8. Conclusions

In summary, for the QES sextic potential with integer $N > 0$ an algebraic polynomial operator $h(x, \partial_x; N)$ that governs the N exact polynomial solutions of its 1-SUSY partner $V_1(x)$ is constructed. These odd-parity solutions are polynomials in the variable x of order $(4N - 3)$, occurring in the form of zero modes. Nor in the x -variable neither in x^2 , the operator $h(x, \partial_x; N)$ is \mathfrak{sl}_2 Lie-algebraic. In the case $N = 0$, the potential V_1 possesses a \mathfrak{sl}_2 hidden Lie algebra, but no exact solutions occur.

At fixed $N > 0$, the potential $V_1(x)$ splits into two additive parts. The first one is polynomial, and it is given by $V^{\text{qes}}(x)$ with a different quantized parameter $N \rightarrow N - \frac{3}{2}$, whereas the second part is a rational function in the variable x^2 . This decomposition represents an important relation between this pair of SUSY partner potentials. For instance, at large distances $|x| \rightarrow \infty$ the rational terms in V_1 vanish as $\frac{1}{x^2}$. This means that the asymptotic behaviour $\sim \exp \left[-\frac{1}{4} x^4 - \frac{1}{2} x^2 \right]$ of the solutions for V_1 is the same as that occurring for V^{qes} . Furthermore, since $V^{\text{qes}}(x)$ admits exact analytical solutions at $N = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$, it follows that for some states with $N > 1$ the

prefactor $F(x)$ in the solutions of $V_1(x)$, $\phi(x) = F(x)\exp\left[-\frac{1}{4}x^4 - \frac{1}{2}x^2\right]$, tends to a polynomial function at large distances $|x| \rightarrow \infty$.

For the lowest principal quantum numbers $n = 0, 1, 2$ of V^{qes} , highly accurate values for the energy (~ 20 s. d.) as a function of $N \in [-1, 3]$ were calculated in the non-algebraic sector of the spectrum. In particular, we determined the critical value $N_c = 0.732953126$ above which tunneling effects, completely absent in the exact analytical solutions, can occur. As for the first two excited states with $N = 0$, compact physically relevant trial functions are constructed. They are used to estimate the corresponding decrease in accuracy, around one order of magnitude, when supersymmetric quantum mechanics is applied on the level of approximate solutions. This serves to determine the applicability of SUSY techniques in the non-algebraic sector of the QES systems.

We plan to study interesting open questions such as the possible presence of instanton-like terms in the non-algebraic sector of $V^{\text{qes}}(x)$, the existence of a hidden Lie algebra of $V_1(x)$ and $V_2(x)$ in a special variable $z = z(x)$ as well as the corresponding SUSY relations within the framework of path integral, namely, at the level of Feynman diagrams.

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Data availability statement

No new data were created or analysed in this study.

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